Energy Landscape of 3D Codes

"We prove that any sequence of local errors mapping a ground state of a [stabilizer Hamiltonian with no string-like logical operators] to an orthogonal ground state must cross an energy barrier growing at least as a logarithm of the lattice size."

On the energy landscape of 3D spin Hamiltonians with topological order
Bravyi, Haah 2011 arXiv: 1105.4159

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Stabilizer Notation

Let $\Lambda = \{1, \ldots, L\}^D$ be a D-dimensional cubic lattice, where sites $u \in \Lambda$ correspond to a finite number of qubits. Define a Hamiltonian:

$$H = -\sum_{a=1}^{M} G_a$$

Where each G_a is a local Pauli operator acting on and $\mathcal{G} = \langle G_1, \ldots, G_M \rangle$ is an abelian group called the stabilizer group of the code. The generators G_a act non-trivially only on the elementary cubes of Λ, which can always be done by coarse-graining.

We assume that H is frustration-free: the ground state ψ_0 of H satisfies $G_a \psi_0 = \psi_0$ for all a. This follows if the generators are independent and $-I \notin \mathcal{G}$.

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Defects, Syndromes, and the Vacuum

Consider a multi-qubit Pauli operator E implementing an error, so that $\psi = E \psi_0$ is an excited eigenstate of H which satisfies $G_a \psi = \pm \psi$ for each a, depending on whether E commutes or anticommutes with G_a.

The ground states of stabilizer Hamiltonians are topologically ordered, in a sense we'll soon define. Borrowing terms from physics, these ordered ground states are called **vacuum** and the flipped generators $G_a \psi = -\psi$ in an excited state are called **defects.**

The set of defects created by applying E to the vacuum is called the **syndrome** of E, and does not depend on the choice of ground state. An excited state with m defects has energy $2m$ above the ground state.

If the action of E on the vacuum creates no defects, then E is either a stabilizer or a logical operator.

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Topological Order

Intuitively, a Hamiltonian is said to have **topological order** if it has a degenerate ground state and different ground states are locally indistinguishable.

To define "locally indistinguishable" requires a length scale L_{TQO} that is bounded as $L_{\text{TQO}} \geq L^\beta$ for some constant β. For any Pauli operator V of size less than L_{TQO} and any distinct ground states $|0\rangle, |1\rangle$ of H we have:

$\langle 1 \mid V \mid 0 \rangle = 0$ and

$\langle 1 \mid V \mid 1 \rangle = \langle 0 \mid V \mid 0 \rangle$

From this definition we see that stabilizer Hamiltonians have topological order, where L_{TQO} corresponds to the code distance, which is typically $\sim L$.

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Locality of Neutral Defect Clusters

A cluster of defects is called **neutral** if it can be created from the vacuum by a single Pauli operator E without creating any other defects. Otherwise it is called **charged**.

Example: In the toric code, flipping a single qubit produces two defects (neighboring plaquettes). There's no Pauli operator that will produce exactly one defect, so a single defect by itself is charged, while a pair of defects created together by some Pauli operator are neutral.

The terminology is an analogy from physics, where conservation of e.g. electric charge means that a set of particles created from the vacuum in the same event will have a total charge of zero, because the vacuum is neutral.

A stronger version of topological order requires that neutral defects can be created locally: if S is a neutral cluster of defects and $C_{\min}(S)$ is the smallest cube that encloses S, then S can be created from the vacuum by a Pauli operator supported on $\mathcal{B}_1(C_{\min}(S))$.

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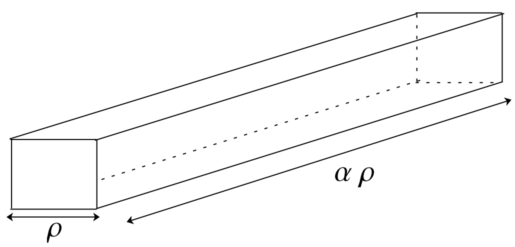
No-Strings Rule



A Pauli operator E is a logical string segment with parameters (ρ, α) if the set of defects S created by applying E to the vacuum satisfies $S \subseteq A_1 \bigcup A_2$ where A_1 and A_2 (called **anchor regions**) are disjoint cubes of linear size ρ at a distance $\alpha\,\rho$ apart from each other.

Definition ("no strings rule"): A code has no string-like logical operators if there exists a constant α such that all logical string segments with aspect ratio greater than α have neutral anchor regions.

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Error Paths and Energy Barriers

A sequence $E_1, \ldots, E_T$ is an **error path** for the logical operator $\overline{P}$ if each E_i is a local Pauli operator and $\overline{P} = E_T \ldots E_2 E_1$.

Applying this sequence of errors to a ground state ψ_0 generates a syndrome history $\{S(t)\}_{t=0,\ldots,T}$, where $S(0) = \emptyset$ and $\psi(T) = \emptyset$ are vacuum, and $S(t)$ are non-empty sets of defects.

The operator $\overline{P}$ has **energy barrier** ω if the syndrome history for any error path for $\overline{P}$ contains a syndrome with at least ω defects.

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Bounds on the Energy Barrier

Both theorems apply to any stabilizer Hamiltonian that obeys the topological order condition and the no-strings rule:

Theorem 1. The energy barrier for any logical operator is at least $c \log L$, where L is the lattice size and c is a constant.

Theorem 2. Let S be a neutral cluster of defects containing a charged cluster $S' \subseteq S$ of diameter r such that there are no other defects within distance R from S'. If $r + R < L_{\text{TQO}}$, then the energy barrier for creating S from the vacuum is at least $c \log R$ for some constant c.

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Coarse-Graining for Theorem 1

For each postitive integer p define a coarse-grained lattice with elementary unit of length:

$$\xi(p) = (10\,\alpha)^p$$

Definition: A syndrome S is **sparse** at level p if it is contained in a disjoint union of level-p elementary cubes which are all at least a distance $\xi(p+1)$ apart from each other, otherwise S is **dense** at level-p.

To prove theorem 1 we will show that a syndrome which is dense at levels $0, \ldots, p$ contains at least p defects, and that logical operators for a code satisfying the no-strings rule are dense to a level of at least $p_{\text{Max}} = \Omega(\log L)$.

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Dense Syndrome Lemma

Lemma. Suppose a syndrome S is dense at all levels $0, \ldots, p$, then S contains at least $p + 2$ defects.

Proof. Let $C_1^{(0)}, \ldots, C_g^{(0)}$ be level-0 elementary cubes containing $S(t)$. Since S is dense at level 0 there exists a pair $C_a^{(0)}, C_b^{(0)}$ such that $C_a^{(0)} \cup C_b^{(0)}$ is contained in a level-1 elementary cube $C_{a,b}^{(1)}$. But S is also dense at level 1 so there must be a another $C_c^{(0)} \nsubseteq C_{a,b}^{(1)}$ and hence $g \geq 3$. Continuing in this way we arrive at $g \geq p + 2$.

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Coarse-Grained Syndrome Histories

Given an error path $E_1, \ldots, E_T$ define a syndrome history $\{S(t)\}_{t=0,\ldots,T}$, where $S(t)$ is the set of defects obtained by applying $E_t \ldots E_1$ to the vacuum.

Define the level p coarse-grained syndrome history as the subsequence of $\{S(t)\}$ consisting of operators which are dense at all levels $0, \ldots, p-1$.

Let $S(t')$ and $S(t'')$ be a consecutive pair of level p syndromes. The product of all single-qubit errors E_j that occured between $S(t')$ and $S(t'')$ is called a level p error operator.

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Localizing Level-p Errors

Informally, if $\xi(p)$ is sufficiently smaller than L_{TQO}, then any error operator at level p is equivalent to a a level p local operator, modulo stabilizers.

Lemma. Let S' and S'' be a consecutive pair of syndromes in the level-p syndrome history, connected by the level-p error E, and let m be the the maximum number of defects in the syndrome history. If $4\,m(2+\xi(p)) \leq L_{\text{TQO}}$ then there exists an error $\tilde{E}$ supported on $\mathcal{B}_{\xi(p)}(S' \bigcup S'')$ such that $E\,\tilde{E}$ is a stabilizer.

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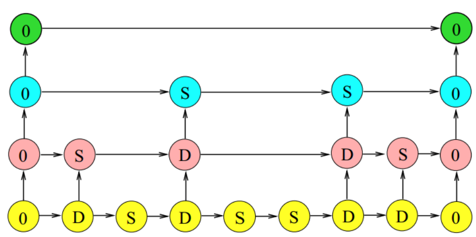
Proof of Theorem 1



Proof of Theorem 1. Given a non-trivial logical operator $\overline{E}$, let $p_{\max}$ be the least value of p for which the level-$p_{\max}$ syndrome history consists of a single level-$p_{\max}$ error E mapping the vacuum to itself. If $4\,m(2+\xi(p)) \leq L_{\text{TQO}}$ then we can apply the above lemma, and since $\tilde{E} = I$ this implies that E must be a stabilizer, which contradicts the assumption that E is non-trivial. Therefore $4\,m(2+\xi(p)) \geq L_{\text{TQO}}$, so $p_{\max} = \Omega\left(\log L_{\text{TQO}}\right) = \Omega(\log L)$.

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Open Questions

How to perform error correction to extract encoded information from these 3D codes?

Is strong self-correction of the kind observed in the 4D toric code possible in 3D?

What about self-correcting memories which are not based on stabilizer Hamiltonians?

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