Random Unitaries give Quantum Expanders

Experience with classical expander graphs suggests that, while finding deterministic constructions of them is difficult, with high probability a random graph of fixed degree greater than 2 is an expander.

Today we will see that randomly choosing the matrices in a completely positive map from the unitary group gives a quantum expander.

Random Classical Expanders

Suppose $G = (V, E)$ has n vertices. For a subset S of V we define the edge boundary of S, ∂S, as the set of edges connecting S to its complement $\overline{S}$:

$$\partial S = \{(u, v) \in E : u \in S \text{ and } v \in \overline{S}\}$$

The edge-expansion parameter for G is defined by:

$$h(G) = \min_{S:|S| \le n/2} |\partial S| / |S|$$

If G is a random d-regular graph and S is a subset of at most $n/2$ vertices, then a typical vertex in S will be connected to roughly $d\,|\overline{S}|/n$ vertices in $\overline{S}$, so $|\partial S| \approx d\ |S|\ \ |\overline{S}|/n$. Since $|\overline{S}|$ is at least $n/2$ we have $h(G) \approx d/2$, independent of n.

Quantum Expanders

A quantum expander $\mathcal{E}(M)$ is a quantum operation (a completely positive, trace preserving map from $\mathbb{C}^{N\times N}$ to itself) with a gapped eigenvalue spectrum ($\mathcal{E}$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{N^2}$ with $\lambda_1 = 1$ and $|\lambda_a| \le 1 - \delta$ for all $a > 1$, for some δ independent of N). The map can be written as:

$$\mathcal{E}(M) = \sum_{s=1}^{D} A(s)^\dagger M\, A(s) \quad , \quad \text{with} \sum_{s=1}^{D} A(s)^\dagger A(s) = \sum_{s=1}^{D} A(s)\, A(s)^\dagger = \mathbb{1}$$

for D that is either constant, or at least small relative to N. The eigenvector corresponding to λ_1 is $\left(1\big/\sqrt{N}\right)\mathbb{1}$. We will consider maps of the above form with:

$$A(s) = \frac{1}{\sqrt{D}}\, U(s)$$

for some unitary matrices $\{U(s)\}$. We can make $\mathcal{E}$ Hermitian by making D even and choosing $U(s)^\dagger = U(s + D/2)$.

Main Result

Define $\lambda_H = 2\sqrt{D-1}\big/D$. If $\mathcal{E}$ is a Hermitian map as described above with the unitaries $U(s)$ choosen randomly from the Unitary group and $D \ge 4$, then for any $\epsilon > 0$ the probability that $|\lambda_2|$ is within ϵ of λ_H approaches unity as $N \to \infty$. In the non-Hermitian case the same result holds with $D \ge 2$.

The statement "randomly chosen from the unitary group" requires some elaboration. It is meant that the unitaries are chosen uniformly at random according to the Haar measure, which respects the group structure. To explain this we will take a short detour into the measure theory of topological groups.

Measure Theory

In a topological space, the Borel sets are those which can be formed from the open sets by countable unions, countable intersections, and relative complements. In the case of the real numbers $\mathbb{R}$ with the euclidean topology, the Borel sets include all intervals.

The reason the Borel sets are defined this way is to provide a starting point for talking about measurable sets. For our purposes a measure is a non-negative function μ which maps Borel sets to $\mathbb{R} \bigcup \{\infty\}$ and satisfies countable additivity:

$$\mu\left(\bigcup_{i\in I} S_i\right) = \sum_{i\in I} \mu(S_i)$$

for any collection of disjoint Borel sets S_i indexed by a countable set I. We also require $\mu(\emptyset) = 0$. The standard measure on the real line is defined by $\mu([a, b]) = b - a$, and sets which are not Borel sets will be considered non-measurable.

Haar Measure

If our topological space also has the structure of a group G, then we want a measure on the Borel sets of G that is invariant with respect to the group action in the following way:

$$\mu(a\,S) = \mu(S)$$

for all Borel sets $S \subseteq G$ and elements $a \in G$. This is called left-invariance; right-invariance would analogously be $\mu(S\,a) = \mu(S)$.

Haar's theorem says that, up to a positive multiplicative constant, there is a unique countably additive nontrivial regular measure μ on Borel sets which is left-invariant and assigns finite measure to every compact set. If G is compact we can choose $\mu(G) = 1$ to set the normalization.

Example Haar Measures

The Haar measure on $(\mathbb{R}, +)$ which assigns $\mu([0, 1]) = 1$ is the standard measure corresponding to $\mu([a, b]) = b - a$ for any interval.

The Haar measure on the non-zero reals under multiplication is defined by:

$$\mu(S) = \int_S \frac{1}{|x|} \, dx$$

For the group $U(n)$ explicit formulas are harder to come by. If we characterize unitary matrices by their eigenvalues $\left(e^{i\,\theta_1}, \ldots, e^{i\,\theta_n}\right)$, then Weyl's integration formula for conjugancy invariant functions f is given by:

$$\int_{U(n)} f(A)\, dA = \frac{1}{(2\pi)^n\, n!} \int_{\theta_1=0}^{2\pi} \ldots \int_{\theta_n=0}^{2\pi} \prod_{j<k} \left| e^{i\theta_j} - e^{i\,\theta_k} \right|^2 f(\theta_1, \ldots, \theta_n)\, d\theta_1 \ldots d\theta_n$$

this formula could be used to determine the measure of sets for which the indicator function satisfies the above conditions on f. In general Haar's theorem only guarantees us the existence of a measure without giving us a way to compute it for explicit sets, and sampling unitaries at random according to the Haar measure is a hard problem.

Main Proof: Trace Method

To upper bound the second eigenvalue of the proposed expander $\mathcal{E}$ we will use the trace of $\mathcal{E}^m$:

$$\operatorname{Tr}(\mathcal{E}^m) = \sum_{a=1}^{N^2} |\lambda_a|^m \geq 1 + |\lambda_2|^m$$

Since $\mathcal{E}^m$ denotes applying $\mathcal{E}$ m-times we can substitute the definition and get:

$$\mathcal{E}^m(M) = \frac{1}{D^M} \sum_{s_1=1}^{D} \ldots \sum_{s_m=1}^{D} U(s_m)^\dagger \ldots U(s_1)^\dagger M\, U(s_1) \ldots U(s_m)$$

To compute the trace we will apply $\mathcal{E}^m$ to the complete orthonormal basis of matrices $M(i, j)$ which have a 1 in the (i, j) position and 0s elsewhere:

$$\operatorname{Tr}(\mathcal{E}^m) = \sum_{i,j} \langle M(i, j), \mathcal{E}^m(M(i, j))\rangle = \sum_{i,j} \operatorname{tr}[M(j, i)\, \mathcal{E}^m(M(i, j))]$$

$$= \sum_{i,j} \operatorname{tr}\left[M(j, i) \left(\frac{1}{D^M} \sum_{s_1=1}^{D} \ldots \sum_{s_m=1}^{D} U(s_m)^\dagger \ldots U(s_1)^\dagger M(i, j)\, U(s_1) \ldots U(s_m)\right)\right]$$

$$= \frac{1}{D^M} \sum_{s_1=1}^{D} \ldots \sum_{s_m=1}^{D} \sum_{i,j} \operatorname{tr}\left[M(j, i)\; U(s_m)^\dagger \ldots U(s_1)^\dagger M(i, j)\, U(s_1) \ldots U(s_m)\right]$$

To manipulate this further we will write it out in terms of indices and use the fact that $M(i, j)_{\mu\nu} = \delta_{\mu i}\, \delta_{\nu j}$. We use $*$ for internal indices that we won't touch, and we use the convention that repeated indices are summed over.

$$M(j, i)_{\mu\nu}\; U(s_m)^\dagger_{\nu *} \ldots U(s_1)^\dagger_{*\sigma}\, M(i, j)_{\sigma\rho}\, U(s_1)_{\rho *} \ldots U(s_m)_{*\mu}$$

$$= \delta_{\mu j}\, \delta_{\nu i}\, U(s_m)^\dagger_{\nu *} \ldots U(s_1)^\dagger_{*\sigma}\, \delta_{\sigma i}\, \delta_{\rho j}\, U(s_1)_{\rho *} \ldots U(s_m)_{*\mu}$$

$$= U(s_m)^\dagger_{i *} \ldots U(s_1)^\dagger_{* i}\, U(s_1)_{j *} \ldots U(s_m)_{* j}$$

$$= \operatorname{tr}\left[U(s_m)^\dagger \ldots U(s_1)^\dagger\right] \operatorname{tr}[U(s_1) \ldots U(s_m)]$$

Therefore we have shown:

$$\operatorname{Tr}(\mathcal{E}^m) = \frac{1}{D^M} \sum_{s_1=1}^{D} \ldots \sum_{s_m=1}^{D} \operatorname{tr}\left[U(s_m)^\dagger \ldots U(s_1)^\dagger\right] \operatorname{tr}[U(s_1) \ldots U(s_m)]$$

this is the quantity whose expectation over the unitary group we must compute.

Lower Bound on λ_2

Since $U(s)^\dagger = U(s+D/2)$ in the Hermitian case, certain choices of $s_1, \ldots, s_m$ will reduce $\mathrm{tr}[U(s_1) \ldots U(s_m)]$ to the trivial trace of the identity matrix by cancelling successive appearances of $U(s)\, U(s+D/2)$. This combinatorical observation allows us to find a lower bound on λ_2 that applies to every map $\mathcal{E}$ of the form we have considered.

The number of ways these cancellations can occur is proportional to the probability that a random walk starting at the root of a degree D Cayley tree returns to the root after m steps. The return probability for this walk is a standard textbook result, and combining this with the fact that $\mathrm{Tr}(\mathcal{E}^m) \geq 1 + N^2 \lambda_2$ it is possible to show the following lower bound:

$$\left|\lambda_2\right| \geq \lambda_H \left(1 - O\left(\frac{\log(\log(N))}{\log(N)}\right)\right)$$

Schwinger-Dyson Equations

To estimate this quantity we will use the invariance of expectations over the unitary group under an infinitesimal change of variables:

$$U(s) \to (1 + i\,\epsilon\, T^a)\, U(s)$$
$$U(s)^\dagger \to U(s)\,(1 - i\,\epsilon\, T^a)$$

where the T^a, $a = 1, \ldots, N^2$ are Hermitian matrices chosen to satisfy:

$$\sum_a T^a_{\mu\nu}\, T^a_{\rho\sigma} = \delta_{\mu\sigma}\, \delta_{\nu\rho}$$

Applying the transformation and keeping only terms up to first order in ϵ:

$$U(s_1) \ldots U(s_m) \to (1 + i\,\epsilon\, T^a)\, U(s_1) \ldots (1 + i\,\epsilon\, T^a)\, U(s_m)$$
$$= U(s_1) \ldots U(s_m) + i\,\epsilon \left\{ T^a\, U(s_1) \ldots U(s_m) + \sum_{j=2}^{m} U(s_1) \ldots U(s_{j-1})\, T^a\, U(s_j) \ldots U(s_m) \right\}$$

Now we will multiply by T^a and take the sum over a. We will assert that the expectation of $\mathrm{Tr}[T^a\, U(s_1) \ldots U(s_m)]$ is unchanged by the transformation above, so the sum over a of the trace of T^a times the term in curly braces above should be zero. Noting that $\sum_a T^a\, T^a = N\, \mathbb{1}$, the first term in curly braces is N times the desired quantity $\mathrm{Tr}[U(s_1) \ldots U(s_m)]$, and so we have the following equality in expectation:

$$\mathrm{Tr}[U(s_1) \ldots U(s_m)] = -\frac{1}{N} \sum_a \mathrm{Tr}\left[T^a \sum_{j=2}^{m} U(s_1) \ldots U(s_{j-1})\, T^a\, U(s_j) \ldots U(s_m)\right]$$

For each j we have a term of the following form, where we are using the convention that repeated indices are summed over:

$$\mathrm{Tr}\left[T^a\ U(s_1)\ldots U(s_{j-1})\,T^a\,U(s_j)\ldots U(s_m)\right] = T^a_{\mu\nu}\ U(s_1)_{\nu *}\ldots U(s_{j-1})_{*\rho}\,T^a_{\rho\sigma}\,U(s_j)_{\sigma *}\ldots U(s_m)_{*\mu}$$

$$= T^a_{\mu\nu}\ T^a_{\rho\sigma}\,U(s_1)_{\nu *}\ldots U(s_{j-1})_{*\rho}\,U(s_j)_{\sigma *}\ldots U(s_m)_{*\mu}$$

$$= \delta_{\mu\sigma}\,\delta_{\nu\rho}\,U(s_1)_{\nu *}\ldots U(s_{j-1})_{*\rho}\,U(s_j)_{\sigma *}\ldots U(s_m)_{*\mu}$$

$$= U(s_1)_{\nu *}\ldots U(s_{j-1})_{*\nu}\,U(s_j)_{\mu *}\ldots U(s_m)_{*\mu}$$

$$= \mathrm{Tr}\left[U(s_1)\ldots U(s_{j-1})\right]\mathrm{Tr}\left[U(s_j)\ldots U(s_m)\right]$$

Therefore we have shown the following equality in expectation:

$$\mathrm{Tr}[U(s_1)\ldots U(s_m)] = -\frac{1}{N}\sum_{j=2}^{m}\mathrm{Tr}\left[\,U(s_1)\ldots U(s_j)\right]\mathrm{Tr}\left[U(s_{j+1})\ldots U(s_m)\,\right]$$

Rest of the Proof

Applying the derivation on the previous slide to a product of traces produces a much longer formula, but the ideas are mostly the same:

$$\begin{aligned}
&E[\mathrm{tr}(U(s_{1,1})U(s_{1,2})\ldots U(s_{1,m_1}))L_2\ldots L_k]\\
&= -\frac{1}{N}\sum_{j=2}^{m_1}\delta_{s_{1,1},s_{1,j}}E[\mathrm{tr}(U(s_{1,1})\ldots U(s_{1,j-1}))\mathrm{tr}(U(s_{1,j})\ldots U(s_{1,m_1}))L_2\ldots L_k]\\
&+\frac{1}{N}\sum_{j=2}^{m_1}\delta_{s_{1,1},s_{1,j+D/2}}E[\mathrm{tr}(U(s_{1,2})\ldots U(s_{1,j-1}))\mathrm{tr}(U(s_{j+1,1})\ldots U(s_{1,m_1}))L_2\ldots L_k]\\
&-\frac{1}{N}\sum_{l=2}^{k}\sum_{j=1}^{m_l}\delta_{s_{1,1},s_{l,j}}E[\mathrm{tr}(U(s_{1,1})\ldots U(s_{1,m_1})U(s_{l,j})U(s_{l,j+1})\ldots U(s_{l,j-1}))L_2\ldots L_{l-1}L_{l+1}\ldots L_k]\\
&+\frac{1}{N}\sum_{l=2}^{k}\sum_{j=1}^{m_l}\delta_{s_{1,1},s_{l,j}+D/2}E[\mathrm{tr}(U(s_{1,2})\ldots U(s_{1,m_1})U(s_{l,j+1})U(s_{l,j+2})\ldots U(s_{l,j-1}))L_2\ldots L_{l-1}L_{l+1}\ldots L_k].
\end{aligned}$$

By recursively feeding terms on the RHS of the Schwinger-Dyson equations back into the LHS we get an infinite series in $1/N$. At each step we look for all the cancelations of the form $U(s)\,U(s+D/2)$, and a detailed analysis of these cancellations show that the recursion terminates at each power of $1/N$ and the series to converges with a controlled error term.

Application: MPS with large Entropy

The matrices $\{A(s)\}$ used in a quantum map can also be associated with a matrix product state as follows:

$$\psi(s_1,\ \ldots,\ s_N) = \mathrm{tr}\left[A(s_1)\ldots A(s_N)\right]$$

where $s_1,\ \ldots,\ s_N$ are spin variables for a 1D quantum system with N sites. The gapped eigenvalue spectrum of $\mathcal{E}$ implies that the state ψ has exponentially decaying correlation functions, but the long-range interactions in the parent Hamiltonian for this expander state imply that the state has large entropy. This construction is discussed further in "Entropy and Entanglement in Quantum Ground States", Hastings (2007).

Also worth noting, from this point of view the dimension of the spins corresponds to the degree D of the expander. Perhaps this is related to the fact that highly entangled 1D construction usually involve a spin dimension $D > 2$ (such as the construction we saw last quarter for a QMA-complete Hamiltonian on a line).