Adiabatic Optimization, Path-Integrals, and Stationary Phase

- Understand the importance of Hamming neighborhood symmetry for QAO success
- Draw connections with a classical random walk MAXSAT algorithm based on an efficient Fourier transform

QAO and Time Evolution

Time dependent Hamiltonian $H(t) = H_f + \Gamma(t)\, H_0$

- $H_0 = -\sum_{i=1}^{n} X_i$
- $H_f = \sum_{z \in \{0,1\}^n} f(z) \quad |z\rangle\langle z|$
- $\Gamma(0) = "\infty"$, $\Gamma(1) = 0$

Infinitesimal Time Evolution $\hat{T}(t) = \exp\{-i\, \Delta\mathrm{t}\, H(t)\}$

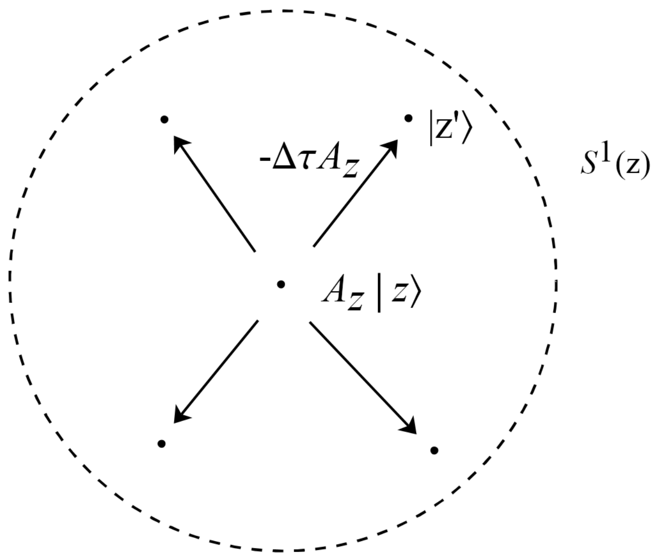
- Imaginary time: $\tau = i\, t$
- Discretize: $\tau_k = k * \Delta\tau$
- Finite Interval: $\hat{T} = \prod_k \exp\{\Delta\tau\, H(\tau_k)\}$
- Trotterize: $\hat{T}(\tau) = \prod_k \exp\{\Delta\tau\, H_f\} \exp\{\Delta\tau\, \Gamma(\tau_k)\, H_0\}$

Off-Diagonal Terms and Hamming Neighborhoods

Hamming metric: $d(z, z'),\ \ z, z' \in \Omega = \{0, 1\}^n$

Hamming Spheres: $S^r(z) = \{z' \in \Omega \ :\ d(z, z') = r\}$

$$\exp\{\Delta\tau H_0\}\ \ |z\rangle = \left\{I - \Delta\tau \sum_i X_i\right\}\ |z\rangle = \ |z\rangle - \Delta\tau \sum_{z' \in S^1(z)} |z'\rangle$$

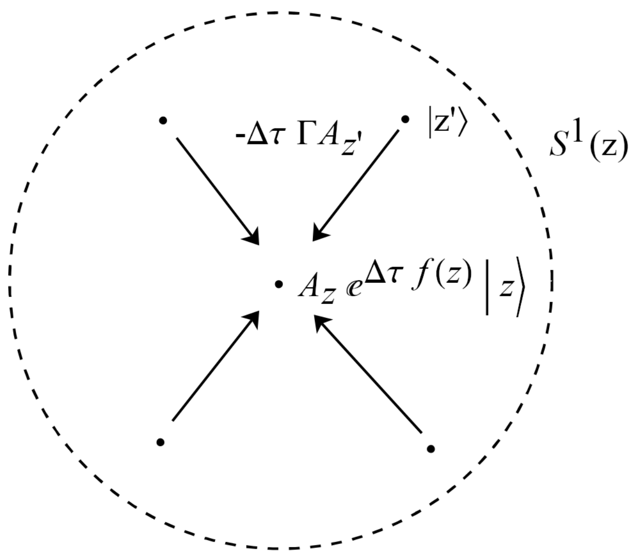


Infinitesimal Time Evolution on a General State

$$\exp\left\{\Delta\tau\, H_f\right\}\ |\,\psi\rangle = \exp\left\{\Delta\tau\, H_f\right\}\sum_{z\in\Omega} A_z\ \ |\,z\rangle\ \ = \sum_{z\in\Omega} A_z\, e^{\beta\, f(z)}\ \ |\,z\rangle$$

$$\left\langle z\,\middle|\,\exp\{\,\Delta\tau\,\Gamma\, H_0\}\exp\left\{\Delta\tau\, H_f\right\}\middle|\,\psi\right\rangle\ =\ A_z\, e^{\Delta\tau\, f(z)}\left\{1\ -\Delta\tau\,\Gamma\sum_{z'\in S^1(z)} A_{z'}\right\}$$

Average amplitude on a sphere: $\langle A\rangle_{S^1(z)}$



Finite Time Intervals

Let $A_z(k)$ be the amplitude of $|\,z\rangle$ after k steps.

$$A_z(k) = A_z(0)\, e^{k\,\Delta\tau\, f(z)} \prod_{k=0}^{1/\Delta\tau} \left\{1 - \Delta\tau\, \Gamma\,(\tau_k)\, \langle A(k)\rangle_{S^1(z)}\right\}$$

$$= A_z(0)\, e^{k\,\Delta\tau\, f(z)} \left\{1 - \Delta\tau \sum_{k=0}^{1/\Delta\tau} \Gamma(\tau_k)\ \langle A(k)\rangle_{S^1(z)} + O\left(\Delta\tau^2\right)\right\}$$

Terms $S^r(z)$ for $r \geq 2$ seem to require maintaining the Trotter approximation to $O\left(\Delta\tau^k\right)$!

Efficiently compute neighborhood averages?

Directed Plateau Search for MAX-k-SAT. Sutton et al, SOCS-2010

"...we develop a surrogate plateau "gradient " function using a Walsh transform of the objective function. This function gives the mean value of the objective function over localized volumes of the search space."

- Fourier basis: $\psi_i(x) = (-1)^{\langle i,x \rangle}$
- Hadamard Transform: $W_{i\,j} = \frac{1}{2^n}\,\psi_i(j)$
- Represent any function : $w = W\,f \;,\; f = \sum w_i\,\psi_i$

Transform can be computed efficiently for SAT!

Efficient Fourier Transform for SAT

Ex: $f(x) = x_0 \vee x_1 \vee \neg x_2 = \begin{cases} 0 & \text{if } x_0 = 1,\ x_1 = 0,\ x_2 = 1 \\ 1 & \text{otherwise} \end{cases}$

- Vector representation: $f = (1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 1\ \ 1\ \ 1)$
- Fourier coefficients are sums of ± 1 with a single term zero-ed out!

Define neg(f) an n-bit string with 1s corresponding to negated bits of f

$$w_j = \begin{cases} \frac{2^n - 1}{2^n} & j = 0 \\ -\frac{1}{2^n}\psi_j(\mathrm{neg}(f)) & j \neq 0 \end{cases}$$

For formulas with C clauses of k literals, w is sparse and computing it takes time $O(2^k C)$.

Averaging over Hamming Spheres

Define sphere matrices: $S^r_{\mathrm{xy}} = \begin{cases} 1 & d(x,\, y) = r \\ 0 & \text{otherwise} \end{cases}$

Walsh functions are eigenvectors: $S^r\,\psi_i = \gamma^r_i\,\psi_i$

$g^r(x)$ = average of all $t_r = \sum_{u=0}^{r} \binom{n}{u}$ states within r of x :

$$g^r(x) = \frac{1}{t_r} \sum_{u=0}^{r} \sum_{y \in S^u(x)} f(y)$$

$$= \frac{1}{t_r} \sum_{u=0}^{r} S^r f(x)$$

$$= \frac{1}{t_r} \sum_{u=0}^{r} \sum_{i} \gamma^r_i\, w_i\, \psi_i(x)$$

Which can be computed in polynomial time!

Plateau Search

Use g to escape plateaus where f is constant

- Even when $f(x) = f(y)$ we may have $g^r(x) \neq g^r(y)$