Matrix Product States

Given a system of quantum spins $s_1, \ldots, s_n$, each having internal dimension D, we associate a set of $k \times k$ matrices $\{A_i(1), \ldots, A_i(D)\}$ with each site i and consider states of the form:

$$\Psi(s_1 \ldots s_n) = \mathrm{tr}[A_1(s_1) \ldots A_n(s_n)]$$

In the translationally invariant case we can use the same set of matrices for each site:

$$\Psi(s_1 \ldots s_n) = \mathrm{tr}[A(s_1) \ldots A(s_n)]$$

The parameter k is called the bond dimension, and it determines the amount of entanglement which can be represented by states of this form. The bond dimension can be thought of intuitively as the amount of "communication" between adjacent sites, and this can be formalized through the picture of MPS as finite state automata.

From Maps to States

We've been considering quantum expanders that act on $k \times k$ density matrices:

$$\mathcal{E}(\rho) = \sum_{s=1}^{D} A(s)^\dagger \rho A(s)$$

Any quantum map of the above form can be associated with a matrix product state for a translationally invariant system. The bond dimension will be k, the internal spin dimension will be D, and the number of sites n can be chosen independently:

$$\Psi(s_1 \ldots s_n) = \mathrm{tr}[A(s_1) \ldots A(s_n)]$$

The connection between the number of Krauss operators and the internal spin dimension is interesting. From Osborne's review on Hamiltonian complexity:

"...it is known that there is a transition in the entanglement structure of the ground state of one-dimensional quantum systems as the local spin dimension changes from 2 to 4. The case of 3-dimensional local spins seems special. Based on this I would be willing to conjecture that the ground-state simulation problem for one-dimensional chains of spin-1/2 degrees of freedom is in **P**."

Parent Hamiltonians

We can construct a Hamiltonian having $\Psi(s_1 \ldots s_n) = \mathrm{tr}[A(s_1) \ldots A(s_n)]$ as its ground state using projectors:

$$H = \sum_i P_{i, i+1, \ldots, i+l}$$

where $P_{i, i+1, \ldots, i+l}$ projects onto the set of states $\Psi_{\alpha,\beta}$ on sites $i, \; i+1, \; \ldots, \; i+l$ defined by:

$$\Psi_{\alpha,\beta}(s_i, s_{i+1}, \ldots, s_{i+l}) = [A(s_i) A(s_{i+1}) \ldots A(s_{i+l})]_{\alpha\beta}$$

The parameter l determines the range of interactions, and for this application we take $l = \log_D(k)$.

Entropy and Correlations

If O_i is a local operator acting on qubit s_i and O_{i+l} is an operator acting on qubit s_{i+l}, then the ground state Ψ of any gapped spin system satisfies:

$$\langle \Psi \mid O_i\, O_{i+l} \mid \Psi \rangle \;=\; e^{-l/\xi} \;+\; \text{const}$$

with the correlation length ξ being a property of the system, independent of the form of the operators O. Since the above relationship suggests that measurements of far apart spins are uncorrelated, it is surprising that gapped spin systems can have large amounts of long-range entanglement.

The idea in this paper is that choosing the matrices $\{A(s)\}$ in the MPS so that the corresponding map is an expander will result in the parent Hamiltonian for the MPS having an energy gap to the first excited state, which implies exponentially decaying correlations. On the other hand, by taking the size l of the projectors in the parent Hamiltonian to be large, the Hamiltonian will have long-range interactions which give rise to a large entanglement entropy.