Quantum Contextuality via Databases and Sheaf Theory

Kevin C. Zatloukal

March 30, 2013
The most fundamental and surprising aspects of quantum mechanics are **contextuality** and **non-locality**.

> Ways in which quantum systems are unlike classical ones.

The goals of this talk are to help us:

1. Understand the distinct *levels* of contextuality.
2. Find simpler means of checking for contextuality in experiments (ideally, computational means).
Bell’s Empirical Model

- Alice and Bob each hold half of an EPR pair:
  \[
  |\phi\rangle^{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)
  \]

- Each decide independently to measure in Z or X basis.
- Call the two outcomes 0 and 1.
- Resulting measurement outcomes ("empirical model"):
Bell’s Theorem

- Einstein would like a theory that satisfies:
  - **locality**: no instantaneous action at a distance
  - **realism**: fixed outcomes for any possible measurement

- That is, he would like a distribution

  \[
  \begin{array}{cccc}
  Z^A & X^A & Z^B & X^B \\
  p_1 & 0 & 0 & 0 & 0 \\
  p_2 & 0 & 0 & 0 & 1 \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  p_{16} & 1 & 1 & 1 & 1 \\
  \end{array}
  \]

  that restricts on each \(AB\) pair to the previous distributions.

- Theorem (Bell): No such distribution \(p_1, \ldots, p_{16}\) exists.

- Usual proof derives contradictory inequalities,
  - but we could just ask Excel’s LP solver to check this.
Next Level Contextuality

- Bell’s empirical model has the *weakest form* of contextuality:
  - Depends on the specific probabilities. If we change some probabilities, the model is locally realistic.
  - Abramsky calls this “probabilistically non-extendability”.
- The next level of contextuality ignores probabilities:
  - Just looks at what outcomes are *possible*.
  - Abramsky calls this “possibilistic non-extendability”.
- For Bell’s empirical model, this is

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>Z</th>
<th>Z</th>
<th>X</th>
<th>X</th>
<th>X</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- This is *possibilistically* extendable. (We’ll see it later.)
Hardy’s Empirical Model

- Hardy described an experiment with possible outcomes:

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>Z</th>
<th>Z</th>
<th>X</th>
<th>X</th>
<th>Z</th>
<th>X</th>
<th>X</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- Unlike Bell’s, this is **not** possibilistically extendable.
- Usual proof derives contradictory boolean formulas.
  - Now we need a SAT solver! Uh oh…
  - Can we do better?
  - Database theory can tell us.
Universal Relations

- Hardy’s empirical model is a set of relations.
  - Relations on attributes $Z^A, X^A, Z^B, X^B$.
- Possibilistic extendability asks if there is a relation on all four attributes that projects to each of the given relations.
- In database theory, this is called a *universal relation*.
  - Important to some foundational aspects of databases.
- Theorem: The most general (i.e. largest) universal relation on $R_1, \ldots, R_k$ is the natural join $R_1 \Join \cdots \Join R_k$.
- If any universal relation exists, the natural join will be one.
  - Compute the natural join and see if it projects correctly.
  - Just type Hardy’s relations into sqlite3.
  - No SAT solver necessary!
- Similar theorem exists for probabilistic databases. We could use these to check probabilistic extendability in the same way.
Using sqlite3 for Bell’s Model

Here is the possibilistic extension:

![SQLite query results]

We can see that restriction to ZZ (1st and 3rd columns) is \{(0, 0), (1, 1)\}. All the others are \{(0, 0), (0, 1), (1, 0), (1, 1)\}. 

Complexity of Finding a Universal Relation

- What if we have a lot of relations? Is this really efficient?
- No, it isn’t. And unfortunately, we can’t do better.
- Theorem (Honeyman, Ladner, Yannakakis): Determining the existence of a universal relation is NP-complete.
- Proof: reduction from 3-colorability
  - One attribute for each vertex.
  - One relation for each edge. Contains pairs of distinct colors.
  - Any row in a universal relation would be a valid coloring, so a universal relation implies 3-colorable.
  - Conversely, if it is 3-colorable:
    - Start with a row that is a valid coloring.
    - Add every permutation of the colors.
    - This will project to the expected relations.
Strong Contextuality

- The *strongest* form of contextuality: no complete set of outcomes (for all measurements) yields a possible subset of outcomes for every compatible set of measurements.
  - Not even a single row whose projections exist in each relation.
- The Hardy model is not strongly contextual:
  - E.g., \((Z^A, X^A, Z^B, X^B) = (0, 1, 1, 0)\) works.
- The GHZ, PR-Box, and Kochen-Specker models are all strongly contextual.
  - As we know, GHZ is realizable in quantum mechanics.
  - The PR-Box model (with measurements U & V for each party):
    
    \[
    \begin{array}{cccc}
    U & U & U & V \\
    0 & 0 & 0 & 0 \\
    1 & 1 & 1 & 1 \\
    V & U & V & V \\
    0 & 0 & 1 & 0 \\
    \end{array}
    \]
  - Proof of strong contextuality: first 3 relations enforce that \(V^A = V^B\), which is not allowed.
Strong Contextuality (cont)

- The other models are too big to draw here.
- Other proofs are also more complicated.
  - Each uses distinct techniques.
- Could try to calculate:
  - We need a SAT solver again.
  - This is clearly NP-complete by earlier construction.
- We should expect any computational procedure is either exponential time or incomplete/unsound.
  - Models that are realizable in QM could be easier…
- Next, we’ll see an incomplete but useful method.
The possibility for contextuality seems related to geometry.

- Databases: *no model* is contextual iff schema is acyclic.
  - I.e., if its fundamental group is trivial.
- Kochen-Specker theorems: conditions on the local geometry that ensure *all models* (of a certain type) are contextual.
- Hence, we should try applying tools from geometry...
Sheaf Theory

- Consider a topological space \( X \).
- Presheaf maps each open set to some other set, in a manner that respects the topology.
  - Usually sets are algebraic: groups, rings, modules, etc.
  - Each element of such a set is called a section.
  - A section of \( X \) (which is an open set) is a global section.
- A sheaf is a presheaf such that, for any open sets \( \{ U_i \} \) covering \( X \), any compatible sections of the \( U_i \) are restrictions of some unique global section.
- In our setting:
  - Open sets are sets of (compatible) measurement labels.
  - Presheaf of distributions on outcomes.
  - Family of sections is an empirical model.
  - Family of sections is compatible iff they satisfy no-signalling.
  - Global sections are locally realistic models.
Sheaf Cohomology

- Sheaf theory provides useful tools, notably sheaf cohomology.
- Gives an obstruction to the existence of a global section.  
  \( H^0 \) contains the global sections and \( H^1 \) the obstructions.
  - Sound but incomplete
  - Čech cohomology is equivalent (usually) and **computable**
- Proves the non-contextuality of GHZ, PR-Box, and some Kochen-Specker models.
- Fails on a Kochen-Specker model that is not realizable in QM
- Conjectured to be complete for QM-realizable models.