Quantum Contextuality via Databases and Sheaf Theory

(summary of arXiv:1208.6416 and arXiv:1111.3620)

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Overview

- The most fundamental and surprising aspects of quantum mechanics are **contextuality** and **non-locality**.
  - Ways in which quantum systems are unlike classical ones.
- The goals of this talk are to help us:
  1. Understand the distinct *levels* of contextuality.
  2. Find simpler means of checking for contextuality in experiments (ideally, computational means).

Bell's Empirical Model

- Alice and Bob each hold half of an EPR pair:

$$|\phi\rangle^{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

- Each decide independently to measure in Z or X basis.
- Call the two outcomes 0 and 1.
- Resulting measurement outcomes ("empirical model"):

	Z	Z
1/2	0	0
0	0	1
0	1	0
1/2	1	1

	Z	X
3/8	0	0
1/8	0	1
1/8	1	0
3/8	1	1

	X	Z
3/8	0	0
1/8	0	1
1/8	1	0
3/8	1	1

	X	X
1/8	0	0
3/8	0	1
3/8	1	0
1/8	1	1

Bell's Theorem

- Einstein would like a theory that satisfies:
  - **locality**: no instantaneous action at a distance
  - **realism**: fixed outcomes for any possible measurement
- That is, he would like a distribution

	Z^A	X^A	Z^B	X^B
p_1	0	0	0	0
p_2	0	0	0	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_{16}	1	1	1	1

  that restricts on each AB pair to the previous distributions.
- Theorem (Bell): No such distribution $p_1, \ldots p_{16}$ exists.
- Usual proof derives contradictory inequalities,
  - but we could just ask Excel's LP solver to check this.

Next Level Contextuality

- Bell's empirical model has the *weakest form* of contextuality:
  - Depends on the specific probabilities. If we change some probabilities, the model is locally realistic.
  - Abramsky calls this "probabilistically non-extendability".
- The next level of contextuality ignores probabilities:
  - Just looks at what outcomes are *possible*.
  - Abramsky calls this "possibilistic non-extendability".
- For Bell's empirical model, this is

Z	Z
0	0
1	1

Z	X
0	0
0	1
1	0
1	1

X	Z
0	0
0	1
1	0
1	1

X	X
0	0
0	1
1	0
1	1

  - This **is** possibilistically extendable. (We'll see it later.)

Hardy's Empirical Model

- Hardy described an experiment with possible outcomes:

Z	Z
0	0
0	1
1	0
1	1

Z	X
0	1
1	0
1	1

X	Z
0	1
1	0
1	1

X	X
0	0
0	1
1	0

- Unlike Bell's, this **is not** possibilistically extendable.
- Usual proof derives contradictory boolean formulas.
  - Now we need a SAT solver! Uh oh...
  - Can we do better?
  - Database theory can tell us.

Universal Relations

- Hardy's empirical model is a set of relations.
  - Relations on attributes Z^A, X^A, Z^B, X^B.
- Possibilistic extendability asks if there is a relation on all four attributes that projects to each of the given relations.
- In database theory, this is called a *universal relation*.
  - Important to some foundational aspects of databases.
- Theorem: The most general (i.e. largest) universal relation on $R_1, \ldots, R_k$ is the natural join $R_1 \bowtie \cdots \bowtie R_k$.
- If any universal relation exists, the natural join will be one.
  - Compute the natural join and see if it projects correctly.
  - Just type Hardy's relations into `sqlite3`.
  - No SAT solver necessary!
- Similar theorem exists for probabilistic databases. We could use these to check probabilistic extendability in the same way

Using sqlite3 for Bell's Model

Here is the possibilistic extension:

```
sqlite> select * from ZZ;
0|0
1|1
sqlite> select * from ZX;
0|0
0|1
1|0
1|1
sqlite> select * from XZ;
0|0
0|1
1|0
1|1
sqlite> select * from XX;
0|0
0|1
1|0
1|1
```

```
sqlite> select distinct ZZ.ZA, XX.XA, ZZ.ZB, XX.XB
   ...> from ZZ join ZX join XZ join XX;
0|0|0|0
0|0|0|1
0|1|0|0
0|1|0|1
1|0|1|0
1|0|1|1
1|1|1|0
1|1|1|1
```

We can see that restriction to ZZ (1st and 3rd columns) is $\{(0,0),(1,1)\}$. All the others are $\{(0,0),(0,1),(1,0),(1,1)\}$.

Complexity of Finding a Universal Relation

- What if we have a lot of relations? Is this really efficient?
- No, it isn't. And unfortunately, we can't do better.
- Theorem (Honeyman, Ladner, Yannakakis): Determining the existence of a universal relation is NP-complete.
- Proof: reduction from 3-colorability
  - One attribute for each vertex.
  - One relation for each edge. Contains pairs of distinct colors.
  - Any row in a universal relation would be a valid coloring, so a universal relation implies 3-colorable.
  - Conversely, if it is 3-colorable:
    - Start with a row that is a valid coloring.
    - Add every permutation of the colors.
    - This will project to the expected relations.

Strong Contextuality

- The *strongest* form of contextuality: no complete set of outcomes (for all measurements) yields a possible subset of outcomes for every compatible set of measurements.
  - Not even a **single row** whose projections exist in each relation.
- The Hardy model is not strongly contextual:
  - E.g., $(Z^A, X^A, Z^B, X^B) = (0, 1, 1, 0)$ works.
- The GHZ, PR-Box, and Kochen-Specker models are all strongly contextual.
  - As we know, GHZ is realizable in quantum mechanics.
  - The PR-Box model (with measurements U & V for each party):

U	U
0	0
1	1

U	V
0	0
1	1

V	U
0	0
1	1

V	V
0	1
1	0

  - Proof of strong contextuality: first 3 relations enforce that $V^A = V^B$, which is not allowed.

Strong Contextuality (cont)

- The other models are too big to draw here.
- Other proofs are also more complicated.
  - Each uses distinct techniques.
- Could try to calculate:
  - We need a SAT solver again.
  - This is clearly NP-complete by earlier construction.
- We should expect any computational procedure is either exponential time or incomplete/unsound.
  - Models that are realizable in QM could be easier...
- Next, we'll see an incomplete but useful method.

Contextuality & Geometry

- The possibility for contextuality seems related to geometry.
- Databases: *no model* is contextual iff schema is acyclic.
  - I.e., if its fundamental group is trivial.
- Kochen-Specker theorems: conditions on the local geometry that ensure *all models* (of a certain type) are contextual.
- Hence, we should try applying tools from geometry. . .

Sheaf Theory

- Consider a topological space X.
- Presheaf maps each open set to some other set, in a manner that respects the topology.
  - Usually sets are algebraic: groups, rings, modules, etc.
  - Each element of such a set is called a *section*.
  - A section of X (which is an open set) is a *global section*.
- A sheaf is a presheaf such that, for any open sets $\{U_i\}$ covering X, any compatible sections of the U_i are restrictions of some unique global section.
- In our setting:
  - Open sets are sets of (compatible) measurement labels.
  - Presheaf of distributions on outcomes.
  - Family of sections is an empirical model.
  - Family of sections is compatible iff they satisfy no-signalling.
  - Global sections are locally realistic models.

Sheaf Cohomology

- Sheaf theory provides useful tools, notably sheaf cohomology.
- Gives an obstruction to the existence of a global section.
  (H^0 contains the global sections and H^1 the obstructions.)
  - Sound but incomplete
  - Čech cohomology is equivalent (usually) and **computable**
- Proves the non-contextuality of GHZ, PR-Box, and some Kochen-Specker models.
- Fails on a Kochen-Specker model that is not realizable in QM
- Conjectured to be complete for QM-realizable models.